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NOTES IN THE THEORY OF
DYNAMIC PROGRAMMING - II:
A FUNCTIONAL EQUATION
ARISING IN ALLOCATION THEORY

by

Richard Bellman

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SUMMARY

~~we derive~~ the structure of the solution of

$$r(x) = \underset{\substack{0 \leq y \leq x \\ \int \int}}{\text{Max}} (g(y) + h(x-y) + f(ay + b(x-y))) ,$$

in the case where g and h are concave is derived.

NOTES IN THE THEORY OF DYNAMIC PROGRAMMING - II:
A FUNCTIONAL EQUATION ARISING IN ALLOCATION THEORY

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§1. Introduction

An equation we have used for illustrative purposes in a number of papers is

$$(1) \quad f(x) = \text{Max}_{0 \leq y \leq x} [g(y) + h(x-y) + f(ay + b(x-y))] \quad .$$

This equation arises in connection with a multi-stage allocation process which we shall describe below.

The purpose of the present note is to complete the description of the solution we have given elsewhere, [1], for the case where g and h are concave, monotone increasing functions. The method of successive approximations that we employ is a very powerful analytic tool for the treatment of functional equations of this general class.

§2. Description of the Process

We are given a resource, x , to divide into two parts, y and $x-y$. From y we obtain a return of $g(y)$; from $x-y$ a return of $h(x-y)$. In so doing we expend a certain amount of the original quantity and are left with a new quantity $ay + b(x-y)$, where $0 < a, b < 1$. This process is now continued indefinitely. How does one allocate at each stage so as to maximize the overall return from the process?

If we let $f(x)$ denote the overall return obtained employing an optimal policy, it is easy to see that an application of the principle of optimality, of. [2], [3], yields the functional equation in (1.1).

§3. The Character of the Solution

The result we wish to prove is

Theorem. Let us assume that

(1) a. $g(x)$ and $h(x)$ are both strictly concave for $x \geq 0$,
monotone increasing, and $g(0) = h(0) = 0$.

b. $g'(0)/(1-a) > h'(0)/(1-b)$, $h'(0) > g'(\infty)$, $b > a$.

Then the optimal policy has the following form:

(a) $y = x$ for $0 \leq x \leq \bar{x}$, where \bar{x} is the root of

(2) $h'(0) = g'(x) + (b-a)g'(ax) + (b-a)ag'(a^2x) + \dots$

(b) $y = y(x)$ for $x \geq \bar{x}$ where $0 < y(x) < x$ and $y(x)$ is the
solution of

(3) $g'(y) - h'(x-y) + (a-b)f'(ay+b(x-y)) = 0$.

Remark: We have given the result only for one set of the possible inequalities connecting $g'(0)$ and $h'(0)$. It is easy to see that the same methods suffice to cover the other cases.

Proof: Let us proceed by successive approximations. For the results we shall tacitly assume concerning the existence and

uniqueness of the solution of (1.1) and the convergence of the successive approximants, we refer to [1], and [3]. The criterion above in (3) for the determination of $y(x)$ is constructive since we can always obtain $f(x)$ directly by means of an iterative process. Here, however, we are primarily interested in the structure of the solution.

Set

$$(4) \quad f_1(x) = \underset{0 \leq y \leq x}{\text{Max}} [g(y) + h(x-y)] .$$

Since, by assumption, $g'(0) > h'(0)$, for small x we have $g'(y) - h'(x-y) > 0$ for $0 \leq y \leq x$. Hence $g(y) + h(x-y)$ is monotone increasing in $0 \leq y \leq x$ and the maximum occurs at $y = x$. As x increases, the equation $g'(y) - h'(x-y) = 0$ will ultimately have a root at $y = x$, and then a root inside the interval $[0, x]$. The critical value of x is given as the solution of $g'(x) - h'(0) = 0$. This equation has precisely one solution, which we call x_1 . For $x \geq x_1$ let $y_1 = y_1(x)$ be the unique solution of $g'(y) = h'(x-y)$. The uniqueness is a consequence of the concavity assumptions concerning g and h .

Thus we have

$$(5) \quad \begin{aligned} f_1(x) &= g(x) , & 0 \leq x \leq x_1 , \\ &= g(y_1) + h(x-y_1) , & x \geq x_1 . \end{aligned}$$

and

$$\begin{aligned}
 (6) \quad f_1'(x) &= g'(x) \quad , \quad 0 \leq x < x_1 \\
 &= [g'(y_1) - h'(x - y_1)] \frac{dy_1}{dx} + h'(x - y_1) = h'(x - y_1), \\
 &\quad \text{for } x > x_1.
 \end{aligned}$$

Since $y_1(x_1) = x_1$, we see that $f_1'(x)$ is continuous at $x = x_1$, and hence, for all values of $x \geq 0$.

Now let us turn to the second approximation

$$(7) \quad f_2(x) = \text{Max}_{0 \leq y \leq x} [g(y) + h(x - y) + f_1(ay + b(x - y))] \quad .$$

The critical function is now $D(y) = g'(y) - h'(x - y) + f_1'(ay + b(x - y))(a - b)$. Since $g'(0) - h'(0) + f_1'(0)(a - b) = g'(0) - h'(0) + g'(0)(a - b) > h'(0) [(1 - a)(1 + a - b)/(1 - b) - 1] > 0$, we see that $D(y)$ is again positive for all y in $[0, x]$ for small x . Hence the maximum occurs in (7) at $y = x$ for small x . As x increases, there will be a first value of x where $D(x) = 0$. This value, x_2 , is determined by the equation $g'(x) = h'(0) + (b - a)f_1'(ax)$. Comparing the two equations

$$\begin{aligned}
 (8) \quad g'(x) &= h'(0) \\
 g'(x) &= h'(0) + (b - a)f_1'(ax) \quad ,
 \end{aligned}$$

we see that $0 < x_2 < x_1$.

Hence the equation for x_2 has the simple form

$$(9) \quad g'(x) = h'(0) + (b - a)g'(ax) \quad .$$

Thus $y = x$ for $0 \leq x \leq x_2$ in (7) and $y = y_2(x)$ for $x \geq x_2$, where $y_2(x)$ is the unique solution of

$$(10) \quad g'(y) = h'(x-y) + (b-a)f'_1(ay+b(x-y)) \quad .$$

Furthermore

$$(11) \quad \begin{aligned} f'_2(x) &= g'(x) \quad , \quad 0 \leq x \leq x_2 \\ &= h'(x-y_2) + b f'_1(ay_2+b(x-y_2)) \quad , \quad x \geq x_2 \quad , \end{aligned}$$

and $f'_2(x)$ is continuous at $x = x_2$.

Comparing (10) with the equation $g'(y) = h'(x-y)$ defining y_1 , we see that $y_2(x) < y_1(x)$. In order to carry out the induction and obtain the corresponding results for all members of the sequence $\{f_n\}$, defined recurrently by the relation $f_{n+1} = \text{Max}_{0 \leq y \leq x} [g(y) + h(x-y) + f_n(ay+b(x-y))]$, we require the essential inequality $f'_2(x) \geq f'_1(x)$. There are three intervals $[0, x_2]$, $[x_2, x_1]$, $[x_1, \infty]$, to examine, each one requiring a separate argument. Using (10) and (11) we have

$$(12) \quad f'_2(x) = \frac{bg'(y_2) - ah'(x-y_2)}{b-a}$$

for $x \geq x_2$. Combining (6) and the equation for y_1 we have

$$(13) \quad f'_1(x) = \frac{bg'(y_1) - ah'(x-y_1)}{b-a}$$

The function $[bg'(y) - ah'(x-y)]/(b-a)$ is monotone decreasing in y for $0 \leq y \leq x$. Since $y_2 < y_1$ we see that $f'_2(x) > f'_1(x)$. This completes the proof for the interval $[x_1, \infty]$. The interval $[0, x_2]$ yields equality. The remaining interval is $[x_2, x_1]$. In this interval, we have

$$(14) \quad \begin{aligned} f'_1(x) &= g'(x) \\ f'_2(x) &= \frac{bg'(y_2) - ah'(x-y_2)}{b-a} \end{aligned}$$

Hence in this interval, since $0 \leq y_2 \leq x$,

$$(15) \quad f'_2(x) \geq \frac{bg'(x) - ah'(0)}{b-a} \geq g'(x) ,$$

since $g'(x) \geq h'(0)$ is a consequence of $g'(y) \geq h'(x-y)$ for $0 \leq y \leq x$ and $0 \leq x \leq x_1$. This completes the proof that $f'_2(x) \geq f'_1(x)$.

We now have all the ingredients of an inductive proof which shows that

$$(16) \quad \begin{aligned} \text{a.} \quad & x_1 > x_2 > \dots x_n > \dots > 0 \\ \text{b.} \quad & f'_1(x) < f'_2(x) < f'_n(x) < \dots \\ \text{c.} \quad & y_1(x) > y_2(x) > \dots \end{aligned}$$

Since $f_n(x)$ converges to $f(x)$, $f'_n(x)$ to $f'(x)$, $y_n(x)$ to $y(x)$ and x_n to \bar{x} , we see that the solution has the indicated form.

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